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ON THE EQUATIONS OF NUMERICAL PREDICTION IN RELATION TO THE NON-DIVERGENCE CONCEPT

by
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The numerical method of short range weather prediction makes use of a system of equations consisting of two equations of motion, the equation of continuity, the equation of heatflow (in adiabatic conditions) and the hydrostatic equation. Taking the horizontal co-ordinates x and y, the reduced pressure ζ ($\zeta = \frac{p}{P}$, where p is the pressure and P the pressure at sea level) and t-the time, as independent variables, the above-mentioned equations may be written as follows:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial H}{\partial x} + lv; \qquad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial H}{\partial y} - lu; \qquad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{P}\frac{\partial \tau}{\partial \zeta} = 0; \qquad (3)$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - \frac{c^2}{PR\zeta}\tau = 0; \qquad (4)$$

$$T = -\frac{\zeta}{R}\frac{\partial H}{\partial \zeta}. \qquad (5)$$

Here,

u and v = wind components along the x and y axis
H = contour height
T = temperature
l = Coriolis parameter
R = gas constant
$\tau = \frac{dp}{dt}$ = individual change of pressure, playing in our system of independent variables the role of vertical motion, being related to it by

$$\tau = \frac{aP}{c^2}\zeta\left(\frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} - gw\right)$$

where w = the vertical component of wind
g = the acceleration of gravity

$$a = \frac{(\gamma_a - \gamma)R}{g},$$

$$c^2 = aR\bar{T},$$

where γ_a = the dry adiabatic lapsrate
γ = the mean lapsrate
$\bar{T}$ = the mean temperature

The system of equations (1)-(5) is solved with respect to the initial and boundary conditions. The τ and T can easily be eliminated from eq.(3)-(5) and the resulting equation system consists of three equations with three unknown

functions u,v, and H. The latter equation system is solved for geostrophic conditions, i.e. it is assumed that horizontal motion of atmospheric disturbances and the temperature field takes place geostrophically, the wind components $u = -\frac{1}{l}\frac{\partial H}{\partial y}$, and $v = \frac{1}{l}\frac{\partial H}{\partial x}$ being determined uniquely by the contour field. For that reason the latter is taken as the initial state. The boundary conditions are usually taken to be that at $\zeta = 0$ $\tau = 0$ and at $\zeta = 1$ $\tau = \frac{gP}{c^2}\frac{\partial H}{\partial t}$.

The validity of the application of the geostrophic approximation is demonstrated in many papers (3,4,5). In particular, I.A. Kibel (4) has given a generalized method for the adaptation of atmospheric motion to the geostrophic motion.

However, the general acceptance of the geostrophic approximation and its application in the equations of numerical prediction has not meant the end of studies on the departures of the actual wind from the geostrophic. Many papers (6,11,12) deal with the estimation of errors resulting from the application of the geostrophic concept in the equations of atmospheric dynamics. In some equations of numerical prediction, the non-geostrophic departures of wind have been accounted for by their approximate values in terms of geostrophic wind.

An alternative approach for taking into account the non-geostrophic effect in the studies on atmospheric processes is to accept the hypothesis of non-divergence of flow, based on the smallness of the horizontal velocity divergence. In this scheme it is assumed that the wind field can be expressed approximately at all levels of the atmosphere by a stream function ψ, so that $u = -\frac{\partial \psi}{\partial y}$, $v = \frac{\partial \psi}{\partial x}$. This approach has been suggested in (1) and (2).

In recent years Bolin (7) and Charney (8), on the basis of experience in numerical prediction with electronic computers, have suggested that the concept of non-divergence be used in the equations of numerical prediction instead of geostrophic approximation as a means of increasing computational stability in forecasting.

Accepting this concept and introducing the stream function into the equations of numerical prediction, these latter can be written for the contour field H at the level of non-divergence* as follows:

$$\frac{\partial \Delta\psi}{\partial t} + (\psi, \Delta\psi) + \beta \frac{\partial \psi}{\partial x} = 0; \qquad (6)$$

$$l\Delta\psi + 2\left[\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2} - \left(\frac{\partial^2\psi}{\partial x \partial y}\right)^2\right] + \beta\frac{\partial\psi}{\partial y} = \Delta H, \qquad (7)$$

where $\beta = \frac{\partial l}{\partial y}$.

This system is solved for a limited flat region using for the initial state the H(x,y) values from the contour map that corresponds to the level of non-divergence.

The method of solution of these equations is as follows: The non-linear equation (7) with known right-hand side is first solved for ψ. This yields the initial stream function corresponding to the initial H field. The solution of eq.(7) is then introduced into eq.(6) and this is solved for ψ, yielding the future distribution of the stream function. This is put back

* According to Landers (10) this level is best represented by the 700 mb surface.

again into eq.(7) and the latter is solved for the now unknown function H, i.e. the future contour field is calculated from the future stream function.

The basic difficulty in the practical application of this model is the initial solution of eq.(7),- the equation of balance -, being a non-linear differential equation of Monge-Ampere type.

Let us examine the individual terms in this equation. If the non-linear terms and the term for the variation of the Coriolis force with latitude are neglected, then the term $\Delta\psi = \frac{1}{l}\Delta H$, relates the contour field with the stream function in a quasi-geostrophic approximation. In other words the eq.(7) transforms, on the given assumption, into the equation of vorticity: $\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$.

The term $\beta\frac{\partial\psi}{\partial y}$ in eq.(7) indicates that the application of the geostrophic approximation for vorticity results in an underestimate of vorticity in westerly flow $\left(\frac{\partial\psi}{\partial y} < 0\right)$ and an overestimate in the easterly flow $\left(\frac{\partial\psi}{\partial y} > 0\right)$.

The non-linear terms $\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2} - \left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2$ can have a pronounced effect only in case of a large curvature combined with large horizontal gradients, as observed in well developed cyclones and anticyclones and, to a lesser extent, in troughs and ridges. In particular, the neglect of the non-linear terms (which occurs with the quasi-geostrophic approximation) leads to an overestimate of vorticity in cyclones and an underestimate in anti-cyclones. Some idea of the actual magnitude of these terms of the Monge-Ampere equation can be obtained from Fig.1.

Let us now discuss some peculiarities of the numerical solution of eq.(7). This equation is of the elliptical type with respect to the unknown function $\psi(x,y)$, if the condition

$$\Delta H - \beta\frac{\partial\psi}{\partial y} > -\frac{l^2}{2}. \tag{8}$$

is satisfied everywhere in and on the boundaries of the region under consideration.

This condition is fulfilled practically everywhere on the particular contour map that corresponds to the level of non-divergence. For the numerical solution of eq.(7), Bolin (7) suggests the relaxation method for one variant of consecutive approximation. The speed of convergence of the iteration process in solving a differential equation by relaxation depends, as is known, on how successfully we have selected the first approximate solution. The latter for eq.(7) can be taken, for instance, in the form $\psi = \frac{H}{l}$, i.e. it is assumed that the stream function is related to the contour field as in the case of geostrophic approximation.

The second variant for approximate solution of eq.(7) might be obtained, for instance, in the following way. Let us rewrite eq.(7) as follows:

$$\Delta q = \frac{2}{l}\left[\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2 - \frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2}\right] - \frac{\beta}{l}\frac{\partial\psi}{\partial y} = Q(x, y) \tag{9}$$

and let us consider it as a Poisson equation for the unknown function $q = \psi - \frac{H}{l}$ with known right hand side, which expresses the vorticity difference assuming a non-divergent flow instead of a geostrophic flow.

Eq.(9) is solved numerically in the following way. We rewrite it in finite differences for n gridpoints, symmetrical with respect to a central point (i,j), for which the solution is sought. On the boundaries of the region under consideration we assume the flow to be geostrophic, i.e. $q = \psi - \frac{H}{l} = 0$ In this way one obtains a system of n linear algebraic equations with n unknowns. The solution of the above system for the unknown q at point (i,j) represents a sum of products of the right hand side of eq.(9) [Q(x,y)], taken at the surrounding n points (including also the central point) by the corresponding weight factors. The right hand side of eq.(9) is computed on the assumption of the geostrophic wind, i.e. $\psi = \frac{H}{l}$.

The values of the individual terms and the whole right hand side of eq.(9), computed from the AT700 map for the morning of Sept. 7, 1953, are given in Fig. 1.*

The method outlined here for the short range numerical prediction of the contour field at the level of non-divergence, based on the solution of eq.(6) and (7) assuming non-divergent flow, can be realized in practice by means of electronic computers as follows:

The initial H_{t_0} at each grid point is read off from an analyzed contour map for the level of non-divergence. These are inserted into eq.(7) and the latter is solved for ψ, yielding the initial stream function ψ_{t_0}. This is now introduced into eq. (6) which forecasts the future stream function:

$$\frac{\partial \Delta \psi}{\partial t} = (\Delta \psi, \psi) - \beta \frac{\partial \psi}{\partial y}.$$

The actual computation of the future stream function is performed by time steps. Introducing ψ_{t_0} into eq.(6), $\frac{\partial \psi}{\partial t}$ and the whole right hand side, containing derivatives of ψ with respect to the coordinates, is computed first for the time instant t_0, i.e. the quantity $\frac{\partial \Delta \psi}{\partial t}$ is calculated for the time instant t_0, assuming $\frac{\partial \psi}{\partial t} = 0$ and $\frac{\partial \Delta \psi}{\partial t} = 0$ at the boundaries of the forecast region.

After computation of $\left(\frac{\partial \Delta \psi}{\partial t}\right)_{t_0}$, the equation

$$(\Delta \psi)_{t_0 + \delta t} = (\Delta \psi)_{t_0} + \left(\frac{\partial \Delta \psi}{\partial t}\right)_{t_0} \delta t, \qquad (10)$$

is applied at all gridpoints, where δt is the "time step" selected, usually according to the dimensions of the grid.

After the numerical value of $\Delta \psi$ at the instant $(t_0 + \delta t)$ is computed for the internal grid points, i.e. the right hand side of eq.(10) is numerically solved for ψ as a Poisson equation, i.e. in the form $(\Delta \psi)_{t_0 + \delta t} = F(x, y)$, the new value of stream function $(\psi)_{t_0 + \delta t}$ itself is calculated for the grid points.

The $\Delta \psi$ and ψ obtained for time instant $t_0 + \delta t$ are now inserted into eq.(10) and the analogous procedure is repeated for the time instant $t_0 + 2\delta t$ etc. etc.

After the $(\psi)_{t_0 + k\delta t}$ and $(\Delta \psi)_{t_0 + k\delta t}$ are found for a given time interval $k\delta t$, where k is the number of time steps, they are introduced into eq.(7).

Considering this equation now as a Poisson equation for the unknown function $H_{t_0 + k\delta t}$

* This figure is not reproduced in this translation.

$$(\Delta H)_{t_0+k\delta t} = l(\Delta\psi)_{t_0+k\delta t} + 2\left[\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2} - \left(\frac{\partial^2\psi}{\partial x \partial y}\right)^2\right]_{t_0+k\delta t} + \beta\left(\frac{\partial\psi}{\partial y}\right)_{t_0+k\delta t}, \qquad (11)$$

with known right hand side, it is solved numerically for H, i.e. the future contour field is calculated.

Next, we derive the system of prognostic equations for a generally baroclinic atmosphere assuming a non-divergent windfield. For this we make use of the initial equations (1)-(5).

Differentiating eq.(1) with respect to y and eq.(2) with respect to x, and subtracting the first from the second, one obtains, as usual, the vorticity equation. Applying now eq.(3) for continuity, we write the resulting equation in the following form:

$$\frac{\partial\Omega}{\partial t} + u\frac{\partial\Omega}{\partial x} + v\frac{\partial\Omega}{\partial y} + \beta v - \frac{l}{P}\frac{\partial\tau}{\partial\zeta} = 0. \qquad (12)$$

where $\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the vertical component of vorticity, and the X-axis is directed toward the east, the y axis toward the north. In addition, we have disregarded Ω in eq.(12) as negligibly small in comparison with l.

Differentiating eq.(1) with respect to x and eq.(2) in respect to y and summing up, one obtains the equation of divergence:

$$\frac{\partial D}{\partial t} + u\frac{\partial D}{\partial x} + v\frac{\partial D}{\partial y} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + 2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \beta u = l\Omega - \Delta H. \qquad (13)$$

where $D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ is the horizontal velocity divergence. No restrictive assumptions have so far been made in respect of wind speed in eq.(12) and (13).

Let us further assume, that the condition

$$D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad (14)$$

is fulfilled approximately at all levels of the atmosphere. The horizontal velocity divergence is thus assumed to be approximately zero everywhere, except for the vorticity equation (term $-\frac{l}{P}\frac{\partial\tau}{\partial\zeta} = lD$). This assumption is based on the estimates of the order of magnitude of terms in the hydrodynamic equations for the atmosphere.

The non-divergence (quasi-solenoidal) assumption for atmospheric motions (eq.14) allows one to introduce a stream function, so that

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x}. \qquad (15)$$

then, $\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \Delta\psi$ and assuming the wind field is non-divergent, eq.(12) and eq.(13) may be written as follows:

$$\frac{\partial\Delta\psi}{\partial t} + (\psi, \Delta\psi) + \beta\frac{\partial\psi}{\partial x} - \frac{l}{P}\frac{\partial\tau}{\partial\zeta} = 0; \qquad (16)$$

$$l\Delta\psi + 2\left[\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2} - \left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2\right] + \beta\frac{\partial\psi}{\partial y} = \Delta H. \qquad (17)$$

In addition, we transform eq.(4) for heat flow. Using eq. (5) and (15) we write it in the following form:

$$\zeta^2\left[\frac{\partial}{\partial\zeta}\left(\frac{\partial H}{\partial t}\right)+\left(\psi,\frac{\partial H}{\partial\zeta}\right)\right]+\frac{c^2}{P}\tau=0. \tag{18}$$

Differentiating eq.(18) with respect to ζ

$$\frac{\partial}{\partial\zeta}\zeta^2\left[\frac{\partial}{\partial\zeta}\left(\frac{\partial H}{\partial t}\right)+\left(\psi,\frac{\partial H}{\partial\zeta}\right)\right]+\frac{c^2}{P}\frac{\partial\tau}{\partial\zeta}=0 \tag{19}$$

and eliminating $\frac{\partial\tau}{\partial\zeta}$ by combining equations (16) and (19) we have:

$$\frac{\partial}{\partial\zeta}\zeta^2\frac{\partial}{\partial\zeta}\left(\frac{\partial H}{\partial t}\right)+\frac{c^2}{l}\Delta\left(\frac{\partial\psi}{\partial t}\right)=-\frac{c^2}{l}\left[(\psi,\Delta\psi)+\beta\frac{\partial\psi}{\partial x}\right]-\frac{\partial}{\partial\zeta}\zeta^2\left(\psi,\frac{\partial H}{\partial\zeta}\right) \tag{20}$$

Eq.(20) contains two unknown functions $\frac{\partial H}{\partial t}$ and $\frac{\partial\psi}{\partial t}$.

The second equation relating these unknowns is obtained by differentiating eq.(17) with respect to time t. The resulting equation:

$$\begin{aligned}\Delta\left(\frac{\partial\psi}{\partial t}\right)+\frac{2}{l}\Big[\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2}{\partial y^2}\left(\frac{\partial\psi}{\partial t}\right)+\frac{\partial^2\psi}{\partial y^2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial\psi}{\partial t}\right)-\\-2\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial\psi}{\partial t}\right)\Big]+\frac{\beta}{l}\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial t}\right)-\frac{1}{l}\Delta\left(\frac{\partial H}{\partial t}\right).\end{aligned} \tag{21}$$

has variable coefficients, however, it is like eq. (20) linear with respect to unknown $\frac{\partial H}{\partial t}$ and $\frac{\partial\psi}{\partial t}$.

Thus, the system of eq.(20) and (21) can be used for the determination of unknown functions $\frac{\partial H}{\partial t}$ and $\frac{\partial\psi}{\partial t}$. In addition to eq.(20) and eq.(21), eq.(17) is used, in analogy with the previous model for the level of non-divergence, to determine the initial ψ function from the given H field. Two boundary conditions for ζ are given in the above equation system.

The solution of the equation system eq.(17), (20) and (21), i.e. the determination of the future distribution of wind field (stream function) and contour field can be realized in practice by means of electronic computers.

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